Exercise 16

Write a trial solution for the method of undetermined coefficients. Do not determine the coefficients.

$$y'' + 3y' - 4y = (x^3 + x)e^x$$

Solution

Since the ODE is linear, the general solution can be written as the sum of a complementary solution and a particular solution.

$$y = y_c + y_p$$

The complementary solution satisfies the associated homogeneous equation.

$$y_c'' + 3y_c' - 4y_c = 0 \tag{1}$$

This is a linear homogeneous ODE, so its solutions are of the form $y_c = e^{rx}$.

$$y_c = e^{rx} \quad \rightarrow \quad y_c' = re^{rx} \quad \rightarrow \quad y_c'' = r^2 e^{rx}$$

Plug these formulas into equation (1).

$$r^2 e^{rx} + 3(re^{rx}) - 4(e^{rx}) = 0$$

Divide both sides by e^{rx}.

$$r^2 + 3r - 4 = 0$$

Solve for r.

$$(r + 4)(r - 1) = 0$$

$$r = \{-4, 1\}$$

Two solutions to the ODE are e^{-4x} and e^x. By the principle of superposition, then,

$$y_c(x) = C_1 e^{-4x} + C_2 e^x.$$

On the other hand, the particular solution satisfies the original ODE.

$$y_p'' + 3y_p' - 4y_p = (x^3 + x)e^x$$

Since the inhomogeneous term is the product of a polynomial and an exponential, the particular solution would be

$$y_p = (Ax^3 + Bx^2 + Cx + D)e^x.$$

e^x already satisfies the complementary solution, though, so an extra factor of x is needed.

$$y_p = x(Ax^3 + Bx^2 + Cx + D)e^x$$